Theorem relating to the development of a reflection coefficient in terms of a small parameter

1981 J. Phys. A: Math. Gen. 14 357

(http://iopscience.iop.org/0305-4470/14/2/011)

J. Phys. A: Math. Gen. 14 (1981) 357–367. Printed in Great Britain

Theorem relating to the development of a reflection coefficient in terms of a small parameter

J Heading
Department of Applied Mathematics, The University College of Wales, Penglais, Aberystwyth, Dyfed, SY23 3BZ, UK

Received 20 June 1980

Abstract. The reflection coefficient R produced by an isotropic medium is expressed in terms of the reflection coefficient r produced by a second isotropic medium known as the 'carrier' medium. At the same time, the field W in the first medium is expressed in terms of an integral equation involving the field w in the carrier medium. The expansion of W and R as power series in terms of a parameter α gives rise to an interesting theorem: the coefficient of α^{n+1} in R contains a large number of multiple integrals of order $n+1$, and *the values of these integrals are mostly equal in pairs.* This pairing of equal contributions to R is a kind of reciprocity between pairs of perturbation terms of order α^{n+1}, the two contributions in each pair being produced by $n+1$ re-radiative processes taking place in differing arrangements.

1. Survey of the problem

In Darwin's work on optics (1924), and Hartree's investigations on ionospheric electromagnetic wave propagation (1929), the following basic assumptions are made: (i) The incident electromagnetic wave propagates undisturbed throughout the medium, as if in free space (this latter free-space medium being designated as the 'carrier' medium); (ii) the individual electrons (bound in Darwin's work, but free in the ionosphere) in the medium, oscillate in sympathy with the electromagnetic field at that point, and through their acceleration re-radiate fresh electromagnetic waves. In this connection, it must be stressed that it is the total field that acts on the electrons, not merely the incident field. Hartree derived the propagation equations for the electromagnetic field in an ionised medium using Hertz oscillators to describe the sources of the fresh electromagnetic waves, these additional waves being responsible for the reflected wave from the medium. White (1942) exploited these ideas in his monograph, while Heading (1953) gave an extended analysis of this theory as applied to plane-stratified media, deriving at the same time some approximate formulae for the field, and for the reflection and transmission coefficients by means of these basic ideas. Westcott, in a series of papers (1962a, b, c, d, 1964), has used these formulae in both isotropic and anisotropic models to show by theoretical and numerical calculations the part that every elementary slice of the medium plays in the reflection process. Heading (1963) produced some general formulae with assumption (i) modified to allow part of the carrier medium to be homogeneous and distinct from free space.

When the medium differs only slightly from free space, the reflection coefficient is susceptible to development as a power series in terms of a small parameter α. Only the coefficients of α and α^2 have been considered by Heading (1953, 1963, 1975), the

0305-4470/81/020357+11\$01.50

coefficients involving integrals of the field throughout the medium. For example, the coefficient of α^2 contains two double integrals, their integrands involving the field throughout the medium together with the two free-space carrier waveforms, one upgoing and one downgoing. Only later did the author realise that these two distinct double integrals are equal in value, though this result was not published, and is not included on pages 113–117 of his text (1975).

The question arises as to whether there are further equalities to be discovered when the coefficients of $\alpha^3, \alpha^4, \ldots,$ are investigated. Moreover, there is the further question as to whether this result for the coefficient of α^2 (and for any others to be discovered in the coefficients of $\alpha^3, \alpha^4, \ldots$) depends on the fact that free-space wave forms are used to 'carry' the various contributions that make up the total field at any point z.

In the present paper, a complete generalisation is made for the propagation of a field W in a medium governed by a general second-order differential equation with variable coefficients, the carrier field being governed by a second general second-order differential equation. Free space lies above and below an inhomogeneous medium confined to the range $a \leqslant z \leqslant b$. An integral equation of the second kind is derived that expresses W in terms of w_1 and w_2 (that is, two special independent solutions of the carrier equation). Additionally, reflection and transmission coefficients are derived that express the reflection coefficient R and the transmission coefficient T for oblique incidence as an integral of W throughout the medium.

The equation for W differs from that for w by means of a parameter α. The development of W and R in terms of α is expressed as a series of perturbation effects, the perturbation field W_n of order α^n giving rise to the following perturbation field W_{n+1} of order α^{n+1}. Each perturbation field of order α^n contains more and more multiple integrals of order n; there are, in fact, 2^n of them. In § 5 a theorem is proved whereby the 2^n contributions to R from W_n, each being of order α^{n+1}, can be separated out into equal pairs when n is odd. When n is even, the same result is true, apart from $2^{n/2}$ special contributions that are excluded. The mathematical nature of these equalities is investigated and a physical interpretation of the fields involved is displayed in a diagram showing the perturbation fields and the equal contributions to R, up to the order α^6.

The theorem and the diagram represent the complete generalisation of the simplest case when $n = 1$ and when free space is used as the carrier medium.

2. The equations under consideration

The propagation of electromagnetic waves in a plane-stratified isotropic plasma is governed by two independent differential equations of the second order,

$$\frac{\mathrm{d}^2 W}{\mathrm{d}z^2} + k^2(C^2 - p(z))W = 0,$$

$$\frac{\mathrm{d}^2 W}{\mathrm{d}z^2} - \frac{2n'}{n}\frac{\mathrm{d}W}{\mathrm{d}z} + k^2(C^2 - p(z))W = 0,$$

where $n^2 = 1 - p$, $C = \cos\theta$, and a prime denotes differentiation with respect to z; see Budden (1961). The former relates to the electric field component perpendicular to the plane of incidence, and the latter to the magnetic field component perpendicular to the plane of incidence. Here $p(z) = X/(1 - \mathrm{i}Z)$ in standard ionospheric notation.

For $z < a$, $z > b$, we shall take $p = 0$, referring to free space.

So that our investigations shall be as general as possible, we shall write either equation in the form

$$W'' + F(z)W' + k^2 G(z)W = 0, \tag{1}$$

where $F(z) = 0$, $G(z) = C^2$ outside the plasma region $a < z < b$.

Whereas Hartree (1929) and Heading (1963) allowed waves to be 'carried' by free-space waves, the object of the present analysis is to allow solutions of equation (1) to be 'carried' by solutions of the independent equation

$$w'' + f(z)w' + k^2 g(z)w = 0, \tag{2}$$

where $f(z) = 0$, $g(z) = C^2$ outside the range $a < z < b$.

A wide range of integral relationships involving W and w has recently been given by Heading (1980); here we focus attention on a special integral expressed in the form of an integral equation, and resembling the method of variation of parameters.

Multiply equation (1) by w, equation (2) by W, subtract and rearrange, giving

$$(wW' - w'W)' + \tfrac{1}{2}(F+f)(wW' - w'W) + \tfrac{1}{2}(F-f)(wW)' + k^2(G-g)wW = 0.$$

The integrating factor is

$$J = \exp\left(\int_a^z \tfrac{1}{2}(F+f)\,\mathrm{d}y\right),$$

yielding

$$[J(wW' - w'W)]' + [\tfrac{1}{2}J(F-f)wW]' = [\tfrac{1}{2}J(F-f)]'wW - k^2 j(G-g)wW.$$

Integration between any two suitable limits finally gives

$$[J(wW' - w'W) + \tfrac{1}{2}J(F-f)wW] = \int \{[\tfrac{1}{2}J(F-f)]' - k^2 J(G-g)\}wW\,\mathrm{d}y. \tag{3}$$

3. The integral equation

We now consider two solutions w_1 and w_2 of equation (2), such that w_1 represents a solution incident from above ($z > b$), and w_2 a solution incident from below ($z < a$). In terms of the respective reflection and transmission coefficients r', t', r, t (a prime attached to these symbols denoting that incidence is from above), we have

$$t' \exp(\mathrm{i}kCz) \leftarrow w_1 \rightarrow \exp(\mathrm{i}kCz) + r' \exp(-\mathrm{i}kCz), \tag{4}$$

$$\exp(-\mathrm{i}kCz) + r \exp(\mathrm{i}kCz) \leftarrow w_2 \rightarrow t \exp(-\mathrm{i}kCz). \tag{5}$$

Write $J(a) = 1$, $J(b) = J_b$. Moreover, let W denote that solution of equation (1) that is incident from below, namely

$$\exp(-\mathrm{i}kCz) + R \exp(\mathrm{i}kCz) \leftarrow W \rightarrow T \exp(-\mathrm{i}kCz).$$

We now place w equal to w_1 and w_2 in equation (3), evaluating this from a to z and from z to b in both cases. We obtain

$$J(w_1 W' - w_1' W) + \tfrac{1}{2}J(F-f)w_1 W + 2\mathrm{i}kCt' = \int_a^z \phi w_1 W\,\mathrm{d}y, \tag{6}$$

$$J(w_2W'-w_2'W)+\tfrac{1}{2}J(F-f)w_2W=-\int_z^b \phi w_2 W\,\mathrm{d}y, \tag{7}$$

$$J(w_1W'-w_1'W)+\tfrac{1}{2}J(F-f)w_1W+J_b 2\mathrm{i}kCT=-\int_z^b \phi w_1 W\,\mathrm{d}y, \tag{8}$$

$$J(w_2W'-w_2'W)+\tfrac{1}{2}J(F-f)w_2W-2\mathrm{i}kCR+2\mathrm{i}kCr=\int_a^z \phi w_2 W\,\mathrm{d}y, \tag{9}$$

where ϕ denotes $\frac{1}{2}[J(F-f)]'-k^2J(G-g)$.

Subtracting equation (9) from equation (7), we obtain the reflection formula

$$R=r+\frac{\mathrm{i}}{2kC}\int_a^b \phi w_2 W\,\mathrm{d}z. \tag{10}$$

Subtracting equation (6) from equation (8), we obtain the transmission formula

$$T=\frac{t'}{J_b}+\frac{\mathrm{i}}{2kCJ_b}\int_a^b \phi w_1 W\,\mathrm{d}z. \tag{11}$$

These are complete generalisations of formulae given by Heading (1963).

Now equations (6) and (7) represent two simultaneous differential–integral equations for W and W'. On account of linearity, W' may be eliminated, giving

$$W(z)=-\frac{2\mathrm{i}kCt'}{D}w_2(z)+\frac{w_2(z)}{D}\int_a^z \phi w_1 W\,\mathrm{d}y+\frac{w_1(z)}{D}\int_z^b \phi w_2 W\,\mathrm{d}y, \tag{12}$$

where $D=J(w_1w_2'-w_2w_1')$. This is an integral equation for W of the second kind; its solution, when substituted into the reflection and transmission formulae (10) and (11), yields R and T respectively. In a sense these formulae are identities, since R and T are known once W is known, without the necessity of carrying out the integrals. On the other hand, progress in the calculation of R can be made without knowing W exactly.

In order to cast equation (12) into a form suitable for the application of the general theorem shortly to be proved, write

$$-2\mathrm{i}kCt'w_1/D=u_1, \qquad -2\mathrm{i}kCt'w_2/D=u_2, \qquad \phi D/(2\mathrm{i}kCt')^2=Y,$$

reducing equations (10) and (12) to

$$R=r+t'\int_a^b Yu_2W\,\mathrm{d}z, \tag{13}$$

$$W=u_2+u_2\int_a^z Yu_1W\,\mathrm{d}y+u_1\int_z^b Yu_2W\,\mathrm{d}y. \tag{14}$$

Here

$$Y=\frac{\{\frac{1}{2}[J(F-f)]'-k^2J(G-g)\}J(w_1w_2'-w_2w_1')}{(2\mathrm{i}kCt')^2}.$$

For the electric field component, $F=f=0$, $J=1$, and we write $G=g+\alpha Q(z)$, $\alpha Q(z)$ being a perturbation effect on the plasma governing the carrier wave. Since the equation is in normal form, the Wronskian $w_1w_2'-w_2w_1'$ is constant, having the value

$-2ikCt'$ below the plasma and $-2ikCt$ above, proving incidentally that $t=t'$. Then

$$Y=\frac{\alpha kQ}{2iCt'}.$$

For the magnetic field component, $F=P'/(1-P), f=p'/(1-p)$, using capital symbols to refer to equation (1). This gives

$$J=[(1-P)(1-p)]^{-1/2},$$

$$Y=\alpha\frac{\left[\frac{1}{2}\left(\frac{Q'(1-p)+Qp'}{[(1-P)(1-p)]^{3/2}}\right)'+\frac{k^2Q}{[(1-P)(1-p)]^{1/2}}\right]}{[(1-P)(1-p)]^{1/2}(2ikCt')^2}(w_1w_2'-w_2w_1').$$

In this expression, although a factor α has been removed, it must be noted that α (through the function P) still exists in the ratio that remains. The theorem to be proved relates to the explicit α that has been removed as a factor.

4. Theorem

If

$$R=r+\alpha t'\int_a^b Su_2W\,dz,$$

$$W=u_2+\alpha u_2\int_a^z Su_1W\,dy+\alpha u_1\int_z^b Su_2W\,dy;$$

if W is developed as a power series in α by successive substitution, yielding 2^n multiple integrals of order n for the coefficient of α^n; and if R is then developed as a power series in α yielding 2^n multiple integrals of order $n+1$ for the coefficient of α^{n+1}, then in the development of R: when n is even there are $2^{n-1}-2^{\frac{1}{2}n-1}$ pairs of equal multiple integrals; when n is odd there are 2^{n-1} pairs of equal multiple integrals. This generalises the case of the coefficient of α^2 in R, for which the two double integrals are equal, giving

$$R\doteqdot r+\alpha t'\int_a^b Su_2^2\,dz+2\alpha^2t'\int_a^b S(z)u_2^2(z)\,dz\int_a^z S(y)u_1(y)u_2(y)\,dy.$$

When the author (1975) wrote pages 115–6 of his text, he did not realise the equality of the two double integrals, so both were evaluated separately in the example given. In that investigation the carrier wave propagated in free space, with $r=0$, $t'=1$, $w_1=\exp(ikz)$, $w_2=\exp(-ikz)$. The present theorem represents a complete generalisation (i) of the carrier waves involved, (ii) of the order of the multiple integrals.

5. Proof of the theorem

To see clearly the nature of the multiple integrals involved, introduce the two operators

$$\mathscr{P}=u_2(z)\int_a^z S(y)u_1(y)\,dy,\qquad \mathscr{Q}=u_1(z)\int_z^b S(y)u_2(y)\,dy,$$

both operating on functions of y. Then

$$W(z) = u_2(z) + \alpha\mathcal{P}W(y) + \alpha\mathcal{Q}W(y).$$

The development in terms of α implied in the theorem is of the form

$$W = \sum_{n=0} \alpha^n W_n,$$

so

$$W_0 = u_2,$$

$$W_1 = \mathcal{P}W_0 + \mathcal{Q}W_0 = (\mathcal{P} + \mathcal{Q})u_2,$$

$$W_2 = \mathcal{P}W_1 + \mathcal{Q}W_1 = (\mathcal{P}\mathcal{P} + \mathcal{P}\mathcal{Q} + \mathcal{Q}\mathcal{P} + \mathcal{Q}\mathcal{Q})u_2 = \sum_4 (\mathcal{P}\mathcal{Q})u_2,$$

where $\Sigma_4\,(\mathcal{P}\mathcal{Q})$ denotes that all four permutations $\mathcal{P}\mathcal{P}$, $\mathcal{P}\mathcal{Q}$, $\mathcal{Q}\mathcal{P}$ and $\mathcal{Q}\mathcal{Q}$ are included; the variable symbol used in the limits of integration in any particular operator is that appearing in the integrand immediately preceding.

Then in R we substitute the form

$$W_n = \sum_{2^n} (\mathcal{P}\mathcal{Q}\ldots)u_2,$$

consisting of 2^n permutations, each being formed by n entries, either $\mathcal{P}$ or $\mathcal{Q}$. Hence, formally, the development of R is

$$R = r + t' \sum_{n=0} \alpha^{n+1} \int_a^b S(z)u_2(z)\,\mathrm{d}z \sum_{2^n} (\mathcal{P}\mathcal{Q}\ldots)u_2. \tag{15}$$

Progress is first made by considering the coefficient of $t'\alpha^2$, namely when $n = 1$:

$$\int_a^b S(z)u_2(z)\,\mathrm{d}z\left(u_2(z)\int_a^z S(y)u_1(y)u_2(y)\,\mathrm{d}y + u_1(z)\int_z^b S(y)u_2(y)u_2(y)\,\mathrm{d}y\right).$$

We assert that these two double integrals are equal. In simpler notation, the structure of the integrands allows us to write the integrals in the forms

$$\int_a^b f(z)\,\mathrm{d}z\int_a^z g(y)\,\mathrm{d}y \qquad \text{and} \qquad \int_a^b g(z)\,\mathrm{d}z\int_z^b f(y)\,\mathrm{d}y, \tag{16}$$

where, for the purpose of this section, the symbols f, g, F, G are distinct from those appearing in equations (1) and (2).

5.1. Proof (i)

The basic idea may be found, for example, on page 6 of the text by Tricomi (1957). The left-hand integral (16) is

$$\int_a^b\int_a^z f(z)g(y)\,\mathrm{d}y\,\mathrm{d}z,$$

the domain of integration being the triangle with vertices

$$(y, z) \equiv (a, a), (b, b), (a, b).$$

The reversal of the order of integration (with z integrated first instead of y) immediately produces the equality.

5.2. *Proof (ii)*

Let $F(z)$, $G(z)$ denote indefinite integrals of $f(z)$ and $g(z)$ respectively. Then the left-hand side of equation (16) is

$$\int_a^b f(z)\,\mathrm{d}z(G(z)-G(a))$$

$$\begin{aligned} &= [F(z)(G(z)-G(a))]_a^b - \int_a^b F(z)g(z)\,\mathrm{d}z \text{ (by parts)} \\ &= F(b)[G(b)-G(a)) - \int_a^b g(z)\,\mathrm{d}z\Big(\int_b^z f(y)\,\mathrm{d}y + F(b)\Big) \\ &= \text{RHS of equation (16).} \end{aligned} \tag{17}$$

We now seek to ascertain whether any similar results apply amongst the 2^n multiple integrals of order $n+1$ appearing in the coefficient of $t'\alpha^{n+1}$, namely amongst

$$\int_a^b S(z)u_2(z)\,\mathrm{d}z \sum_{2^n} (\mathscr{P}\mathscr{Q}\ldots)u_2.$$

Equality exists between any two multiple integrals when the order of integration in one of them is completely reversed. The proof is by induction, since the result is valid when $n=1$.

The result being valid for double integrals, let an integral of order N be the highest-order integral for which the result is valid, namely

$$\int_a^b f(z)\,\mathrm{d}z \int_a^z g(y)\,\mathrm{d}y \int_y^b \ldots \int_w^b h(v)\,\mathrm{d}v \int_a^v j(u)\,\mathrm{d}u \tag{18}$$

$$= \int_a^b j(z)\,\mathrm{d}z \int_z^b h(y)\,\mathrm{d}y \int_a^y \ldots \int_a^w g(v)\,\mathrm{d}v \int_v^b f(u)\,\mathrm{d}u. \tag{19}$$

On the left-hand side, the limits of integration (a as the lower limit, b as the upper limit) correspond to an arbitrary permutation. On the right-hand side, the integrand is the reverse of that in equation (18); the limits from a to b still stand on the left; the remaining integral signs are reversed, such that if in equation (18) b is the upper limit (or a the lower limit) in the Mth integral of the sequence ($M \neq 1$), then a is the lower limit (or b the upper limit) in the $(N-M+2)$th integral in equation (19).

Replace $j(u)$ by $j(u)\int_u^b l(t)\,\mathrm{d}t$ in (18), thereby increasing the order of the multiple integrals (18) and (19) by one. Then equation (19) becomes

$$\int_a^b j(z)\,\mathrm{d}z \int_z^b l(t)\,\mathrm{d}t \int_z^b h(y)\,\mathrm{d}y \int_a^y \ldots \int_z^b f(u)\,\mathrm{d}u$$

$$= \int_a^b \Big(j(z)\int_z^b h(y)\,\mathrm{d}y \int_a^y \ldots \int_v^b f(u)\,\mathrm{d}u\Big)\,\mathrm{d}z \int_z^b l(t)\,\mathrm{d}t. \tag{20}$$

Denote the function in square brackets by $L(z)$. Thus equation (20) equals

$$\int_a^b L(z)\,dz \int_z^b l(t)\,dt$$

$$= \int_a^b l(z)\,dz \int_a^z L(t)\,dt \qquad \text{using (17)}$$

$$= \int_a^b l(z)\,dz \int_a^z j(t)\,dt \int_t^b h(y)\,dy \int_a^y \ldots \int_v^b f(u)\,du,$$

the result now being valid for multiple integrals of order $N+1$. There can therefore be no largest integer N, implying the identity of equations (18) and (19) for all N.

We shall define integral (19) to be the dual of integral (18). Then the dual of any individual integral

$$\int_a^b S(z)u_2(z)\,dz(\mathscr{P}\mathscr{Q}\ldots\mathscr{Q}\mathscr{P})u_2, \tag{21}$$

where $(\mathscr{P}\mathscr{Q}\ldots\mathscr{Q}\mathscr{P})$ denotes any permutation of the operators $\mathscr{P}$ and $\mathscr{Q}$ amongst the n positions, is

$$\int_a^b S(z)u_2(z)\,dz(\mathscr{Q}\mathscr{P}\ldots\mathscr{P}\mathscr{Q})u_2. \tag{22}$$

Here, the second permutation is obtained from the first by reversing the symbols, and changing $\mathscr{P}$ to $\mathscr{Q}$ and $\mathscr{Q}$ to $\mathscr{P}$, namely $\mathscr{P}$ (or $\mathscr{Q}$) in the rth position is replaced by $\mathscr{Q}$ (or $\mathscr{P}$) in the $(n-r+1)$th position. Clearly the integrands have been reversed in this process, and the limits changed in exactly the same way as in equations (18) and (19). Hence the theorem implies the equality of the integrals (21) and (22).

There may be some self-dual integrals, when the permutations in integrals (21) and (22) are identical; for example, when $n=4$,

$$\int_a^b S(z)u_2(z)\,dz(\mathscr{P}\mathscr{Q}\mathscr{P}\mathscr{Q})u_2.$$

When n is odd, there can be no self-dual integrals, for if the permutation in integral (21) contains $\mathscr{P}$ in the central position (that is, the $\frac{1}{2}(n+1)$th position), its dual contains $\mathscr{Q}$ in the same position, so the permutations are distinct though the integrals are equal. Thus there are $2^n/2 \equiv 2^{n-1}$ pairs of equal multiple integrals. When n is even, an integral is self-dual if $\mathscr{P}$ (or $\mathscr{Q}$) occurs in the rth position $(1 \leqslant r \leqslant \frac{1}{2}n)$, and $\mathscr{Q}$ (or $\mathscr{P}$) occurs in the $(n-r+1)$th position $(1 \leqslant r \leqslant \frac{1}{2}n)$, namely, the first $\frac{1}{2}n$ entries in the permutation are arbitrary, but the remaining $\frac{1}{2}n$ entries are then determined. Thus the number of self-dual integrals is $2^{n/2}$, so the number of equal dual pairs is

$$\tfrac{1}{2}(2^n - 2^{n/2}) \equiv 2^{n-1} - 2^{(n/2)-1}.$$

Table 1 shows the numbers involved:

These results form the generalisation of the simple result when $n=1$.

No such theorem exists for the development of the transmission coefficient T, which from equation (11) contains the integral

$$\int_a^b S u_1 W\,dz. \tag{23}$$

Table 1.

n	Number of equal dual pairs	Number of self-dual integrals
1	1	0
2	1	2
3	4	0
4	6	4
5	16	0
6	28	8
7	64	0
8	120	16
9	256	0
10	496	32

When W is substituted, an integral such as (21) is produced, but with u_1 replacing the first u_2. If the operators are reversed, no equivalent to (22) exists; that is, a dual exists, but it does not form part of the expansion of T, so the theorem on equal pairs is no longer applicable.

For if the integral (23) terminates with $\mathcal{Q}u_2$, then its dual in the sense of expression (19) would commence with the integrand

$$u_2^2 \int_a^z u_1 \ldots,$$

But since u_1 *must* precede the first integral, no possibility exists for this dual to exist in the development of T.

If the integral (23) terminates with $\mathcal{P}u_2$, namely

$$u_2 \int_a^z u_1 u_2 \ldots,$$

its dual in the sense of (19) would commence with

$$u_1 u_2 \int_z^b u_2 \ldots$$

written so that u_1 occurs first. But this is an impossibility, since u_2 must not precede an integral containing the limit b.

Hence the development of T contains no dual of any integral. Pairs being absent, the theorem applies only to the reflection coefficient.

6. Physical interpretation

If (2) is the simplest oblique-incidence free-space equation

$$w'' + k^2 C^2 w = 0,$$

we have $r = r' = 0$, $t = t' = 1$, with $w_1 = \exp(\mathrm{i}kCz)$, $w_2 = \exp(-\mathrm{i}kCz)$ as the carrier waves; u_1 and u_2 contain the same waveforms. W consists of a series of perturbation fields produced by the iterative process, either propagating upwards or downwards obliquely as in free space. In fact, $W_0 = w_2$, this original incident wave passing right through the medium, yielding the first perturbed field W_1, consisting of two parts throughout the medium, $\alpha\mathscr{P}w_2$ being upgoing, and $\alpha\mathscr{Q}w_2$ downgoing. Each of these in turn gives rise to two more perturbation fields of order α^2, and so on. Finally, every contribution to the overall perturbation field of all orders of magnitude gives rise to a contribution to the reflection coefficient by substitution into equation (10) or equation (15).

The same observations may be made when the carrier field is governed by the general equation (2). For want of a better nomenclature, we shall still refer to w_2 (or u_2) as upgoing (namely, with the incident wave below), and to w_1 (or u_1) as downgoing.

The field W is illustrated in figure 1, where every straight line segment represents a field throughout the medium. W commences with w_2 at the bottom. Above this are illustrated the two perturbation fields $\mathrm{O}(\alpha)$, $\mathscr{P}w_2$ marked upgoing and $\mathscr{Q}w_2$ downgoing. At the perturbation level $\mathrm{O}(\alpha^2)$, four fields are induced, two arising from each $\mathrm{O}(\alpha)$ contribution.

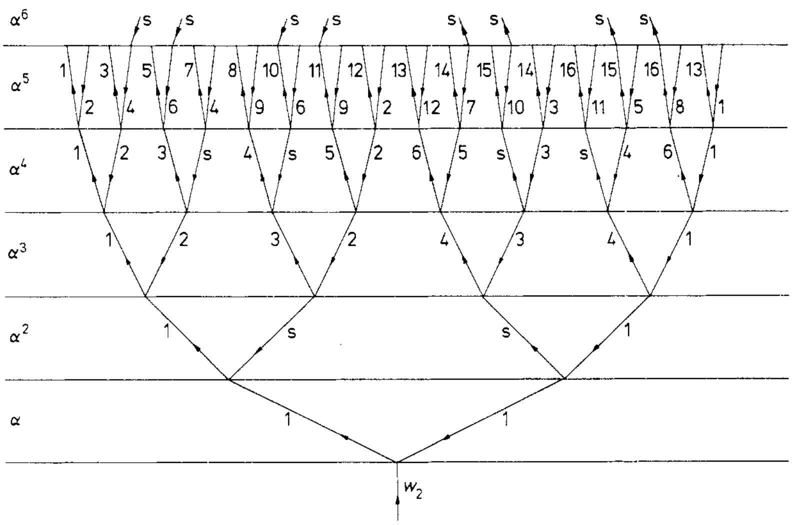

Figure 1. Diagram illustrating the production of various perturbation terms up to order six, and the equalities that they produce in the reflection coefficient.

Any contribution in equation (15), appearing in the integrand from right to left, corresponds in the diagram to a line segment derived from a path from bottom to top, a $\mathscr{P}$ operator corresponding to upgoing (a segment drawn to the left), and a $\mathscr{Q}$ operator to downgoing (a segment drawn to the right). When integrated appropriately throughout the medium, every perturbation in W yields a contribution to R.

The theorem proved in § 5 has shown that equality of contributions to R applies throughout every perturbation band (namely, of order α^{n+1}). In the diagram, dual pairs exist throughout the 32 perturbation fields of $\mathrm{O}(\alpha^5)$; such dual pairs give rise to equal contributions in R, the pairs being decided upon by tracing their origin from w_2 at the base. Thus the permutations

$$\mathscr{P}\mathscr{Q}\mathscr{P}\mathscr{P}\mathscr{Q} \quad \text{and} \quad \mathscr{P}\mathscr{Q}\mathscr{Q}\mathscr{P}\mathscr{Q}$$

(marked 9) yield dual integrals and hence equality in R, a property that we may term reciprocity. (These two cases are up–down–up–up–down and up–down–down–up–down respectively, or in the diagram, left–right–left–left–right and left–right–right–left–right.) Each perturbation band yields such pairs (and also self-dual integrals when the order is even). These are marked in the diagram with corresponding numbers throughout each perturbation band, the symbol s being used to denote self-dual cases.

We have indicated the eight self-dual or self-reciprocal cases in the band $O(\alpha^6)$. In keeping with our calculations in § 5, all the eight perturbation fields of order α^3 give rise to one self-dual field each of the order α^6. This feature, and other phenomena relating to reciprocity and self-reciprocity throughout the perturbation bands may be traced throughout figure 1.

It is this that represents the complete generalisation of the simplest case exhibited in the lowest band.

References

Budden K G 1961 *Radio Waves in the Ionosphere* (Cambridge: CUP)
Darwin C G 1924 *Phil. Trans. Camb. Phil. Soc.* **23** 137–67
Hartree D R 1929 *Proc. Camb. Phil. Soc.* **25** 97–120
Heading J 1953 *PhD Thesis* University of Cambridge
—— 1963 *J. Res. NBS* **67D** 65–77
—— 1975 *Ordinary Differential Equations, Theory and Practice* (London: Elek Science)
—— 1980 *J. Plasma Phys.* To be published
Tricomi F G 1957 *Integral Equations* (New York: Interscience)
Westcott B S 1962a *J. Atmos. Terr. Phys.* **24** 385–99
—— 1962b *J. Atmos. Terr. Phys.* **24** 619–31
—— 1962c *J. Atmos. Terr. Phys.* **24** 701–13
—— 1962d *J. Atmos. Terr. Phys.* **24** 921–36
—— 1964 *J. Atmos. Terr. Phys.* **26** 341–50
White F W G 1942 *Electromagnetic Waves* (London: Methuen)